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COMMENT

Energy and attractors in parallel Potts dynamics

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Abstract. We derive an energy function for multibit threshold automata networks (e.g. the Potts model) updated in parallel, and show that their dynamics admit as attractors fixed points and limit cycles of length 2. We compare this result with similar known results on networks of binary automata (e.g. the Ising model).

Automata networks have been enjoying growing attention as tools for modelling computation, learning, optimisation, and the complexity and organisation emerging from the iteration of simple operations between simple elements (see, e.g., Bienenstock *et al* 1986 and Denker 1986). We focus here on networks with dynamics governed by majority rules. Such networks are nets of interconnected elements, where each element (or site) i has an internal state $x_i = 1, \ldots, p$ and interacts in discrete time steps with other elements j from some neighbourhood v_i of i. In turn, it updates its own state according to a majority rule whereby each site 'aligns' its state to the most prevailing state in the neighbourhood.

The dynamical behaviour depends crucially on whether the sites are updated sequentially or synchronously. In sequential updating, the majority-rule dynamics corresponds to a Monte Carlo evolution of a lattice system at zero temperature, and the system ends up in a (generally local) minimum of the Potts energy:

$$E_{\text{seq}}(\{x_i^t\}) = -\tfrac{1}{2}J \sum_i \sum_{j \in v_i} \delta(x_i^t, x_j^t) \tag{1}$$

where J is a coupling constant and the Kronecker δ is 1 when its two arguments are equal and 0 otherwise. For $p = 2$, E reduces to the Ising energy (up to an additive constant).

When all the sites are updated in parallel, the functional given in (1) is no longer monotonously decreasing. On the other hand, the following 'synchronous' energy:

$$E_{\text{syn}}(\{x_i^t\}, \{x_i^{t-1}\}) = -\tfrac{1}{2}J \sum_i \sum_{j \in v_i} \delta(x_i^t, x_j^{t-1}) \tag{2}$$

never increases under the dynamics; in other words, it is a Lyapunov function. To show in a general way that E_{syn} is indeed a Lyapunov function, we consider a network defined on an arbitrary undirected finite graph, where the connectivity number can vary from site to site, with values x_i evolving under the following parallel dynamics:

$$x_i^{t+1} = \max\{s; \operatorname{card}\{x_j^t = s; j \in v_i\} = \max_{1 \leq r \leq p}\{x_j^t = r; j \in v_i\}\}. \tag{3}$$

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This formula simply means that at each time step, each site will assume the state most represented in the neighbourhood. The parallel nature of the dynamics is expressed in the fact that all the sites i update their state as a function of the *unupdated* states of their neighbours, all taken at the same time t.

Ties in the determination of the most represented state (i.e. in the inner max function in (3)) are broken by arbitrarily choosing the state with the higher s value among the most represented (hence the outer max function in the formula). This way of breaking ties favours high magnetisation. An alternative way consists of modifying the evolution rule by first partitioning the graph in two sets of sites and breaking ties in such a way that the highest value s is in one set but the lowest is in the other set.

Expressing E_{syn}, given by (2), in the form

$$E_{\text{syn}} = -\tfrac{1}{2}J \sum_i \operatorname{card}\{x_j^{t-1} = x_i^t; j \in v_i\} \tag{4}$$

and replacing x_i^t by its value from the update rule (3), we can write E_{syn} in a form that depends on one time step $(t-1)$ only:

$$E_{\text{syn}} = -\tfrac{1}{2}J \sum \max\{s; \operatorname{card}\{x_j^{t-1} = s; j \in v_i\} = \max_r \operatorname{card}\{x_j^{t-1} = r; j \in v_i\}\}. \tag{5}$$

The monotonous decrease of E is made evident by considering the difference:

$$\begin{aligned}\Delta_t E &= \sum_i \left(-\sum_{j\in v_i} \delta(x_i^t, x_j^{t-1}) + \sum_{j\in v_i} \delta(x_i^{t-2}, x_j^{t-1}) \right) \\ &= -\sum_i \left(-\operatorname{card}\{x_j^{t-1} = x_i^t; j \in v_i\} + \operatorname{card}\{x_j^{t-1} = x_i^{t-2}; j \in v_i\}\right).\end{aligned} \tag{6}$$

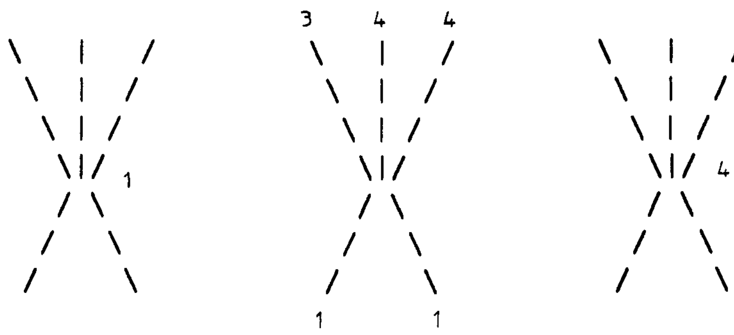
But by definition of the updating rule,

$$\operatorname{card}\{x_j^{t-1} = x_j^t; j \in v_i\} \geqslant \operatorname{card}\{x_j^{t-1} = x_i^{t-2}; j \in v_i\} \tag{7}$$

and therefore:

$$\Delta_t E \leqslant 0. \tag{8}$$

This quantity is now always non-positive: from the definition of the evolution rule (3), the sum with the minus sign is larger than or equal to that with the plus sign. Notice, however, that E_{syn} written as (4) can reach its minimum and remain there for different configurations during transient steps before the attractor is reached. This can be seen in the following example, which shows three successive steps of a site with five neighbours:



$t-2$ $\qquad$ $t-1$ $\qquad$ t

We have in this example:

$$\operatorname{card}\{x_j^{t-1} = x_i^t = 4\} = \operatorname{card}\{x_j^{t-1} = x_i^{t-2} = 1\} = 2 \tag{9}$$

and the ith term of $\Delta_t E$ in (8) vanishes, although the dynamics is still in a transient phase: $x_i^{t-2} \neq x_i^t$.

Adding a 'magnetic field' of strength $1/p$ in the down direction defines a new energy:

$$E^* = E - \frac{1}{p} \sum_i (x_i^t + x_i^{t-1}) \tag{10}$$

which is now a Lyapunov function that reaches its minimum for a unique 'point in phase-space' (sets of configurations $\{x_i\}$ taken at two successive time steps). In the 'partitioned' tie breaking method, the external magnetic field should point down at those sites where s is chosen large but point up for the low-s choices. Indeed, the difference is

$$\Delta_t E^* = \Delta_t E - \frac{1}{p} \sum_i (x_i^t - x_i^{t-2}) \tag{11}$$

with the ith term being

$$(\Delta_t E^*)_i = (\Delta_t E)_i - (1/p)(x_i^t - x_i^{t-2}). \tag{12}$$

If $x_i^t \neq x_i^{t-2}$ and $(\Delta_t E)_i = 0$ (i.e. the tie case), then, with the magnetic field, we must have $x_i^t > x_i^{t-2}$, and thus $(\Delta_t E^*)_i < 0$. If, on the other hand, $x_i^t \neq x_i^{t-2}$ but $(\Delta_t E)_i \neq 0$, we get in a similar way $(\Delta_t E)_i \leqslant 1$ and therefore

$$(\Delta_t E)_i \leqslant 1 - (1/p)(1-p) = -1/p < 0. \tag{13}$$

We conclude that in all cases

$$\Delta_t E^* \leqslant 0 \qquad \text{iff } x_i^{t-2} \neq x_i^t. \tag{14}$$

We also conclude from the very existence of the Lyapunov function E^* that for all initial configurations on a finite graph, the parallel dynamics admits for attractors either fixed points or two-cycles.

Let us now compare this result with what is known in the special case $p = 2$, (the Ising model). Using spin-glass couplings J_{ij} that can depend on the site pairs (with $J_{ij} = J_{ji}$), we found (Goles and Vichniac 1986b) that the $p = 2$ synchronous energy can be written as:

$$E_{\text{syn}}(\boldsymbol{x}^t, \boldsymbol{x}^{t+1}) = -\sum_{i=1}^{N} \left| \sum_{j=1}^{N} J_{ij} x_j^t \right| = -\|\mathbf{J}\boldsymbol{x}^t\|_1 \tag{15}$$

where the values x_i form the components of a vector $\boldsymbol{x}$, and where $\|\cdot\|_1$ is the usual 1-norm (in the 'Manhattan metric') in $\mathbb{R}^N$. This norm, of course, does not apply on the network itself, but on its configuration space. It contrasts with the Lyapunov function of the seqential case (the Hopfield energy), which can be written as a scalar product in the same space:

$$E_{\text{syn}}(\boldsymbol{x}^t) = -\tfrac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} J_{ij} x_i^t x_j^t = -\tfrac{1}{2} \langle \boldsymbol{x}^t, \mathbf{J}\boldsymbol{x}^t \rangle. \tag{16}$$

Other functions, such as E_{syn}, involving arguments taken at two successive time steps have been used in the context of neural networks (Goles 1983, Goles *et al* 1985) and as invariants for reversible rules of the Q2R type (Pomeau 1984, Goles and Vichniac 1986a).

In both the Ising and Potts cases, the geometric interpretation of the energy is (up to an additive constant) the total length of the interfaces between the various clusters. Starting the deterministic (i.e. $T=0$) dynamics out of an initially random lattice (T infinite) describes a quenching process with the competitive growth of clusters of p different kinds. It is known that in the Ising case ($p=2$), the parallel updating leads to unwanted oscillations: neighbouring spins in opposite states 'oversmart' themselves: they simultaneously flip their states in an attempt to align with their neighbours. The interfaces between clusters do not decrease. They can actually increase, forming, in the case of the square lattice, oscillating checkerboard patterns (Vichniac 1984, Hayes 1984). Furthermore, it is also known that the oscillations can be damped if *numerical* intermediate states are introduced between the two Ising states (Vichniac 1984). The present work shows that, somewhat surprisingly, when *physical* states are added at each site in the form of the Potts model (with $p>2$), the oscillations are *not* necessarily damped and their maximum period is again 2. The oscillations and their period 2 are attributes of the *synchronous* updating scheme, not of the number of physical states at each site.

Let us finally point that the dynamics defined by (3) decreases the value of the energy in a steepest descent fashion. On the other hand, slower dynamics (for instance 'voting' for the *second* most represented state) are more complex: in one dimension, they are found (Goles 1989) to be characteristic of 'class 4' behaviour of Wolfram's taxonomy (Wolfram 1984).

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